in Section 7. Its average computation times are uniformly lower than those of the other algorithms, as shown in the final section.

An LAP with costs $c[i,j]$ $(i,j=1 \dots n)$ can be formulated as a linear program:

$$\begin{aligned} &\min \ \Sigma_{ij}\, c[i,j] \cdot x[i,j] \\ &\text{subject to} \\ &\quad \Sigma_j\, x[i,j]=1 \qquad (i=1 \dots n), \\ &\quad \Sigma_i\, x[i,j]=1 \qquad (j=1 \dots n), \\ &\quad x[i,j] \geq 0 \qquad (i,j=1 \dots n). \end{aligned}$$

The dual problem is:

$$\begin{aligned} &\max \ \Sigma_i\, u[i]+\Sigma_j\, v[j] \\ &\text{subject to} \\ &\quad c[i,j]-u[i]-v[j] \geq 0 \qquad (i,j=1 \dots n). \end{aligned}$$

With the dual variables $u[i]$ and $v[j]$ the reduced costs are $c[i,j]-u[i]-v[j]$ $(i,j=1 \dots n)$. So, the dual problem is to find a reduction of the costs matrix with maximum sum and non-negative reduced costs.

In the following, indices i and j refer to rows and columns respectively; $x[i]$ is the column index assigned to row i and $y[j]$ the row index assigned to column j, with $x[i]=0$ for an unassigned row i and $y[j]=0$ for an unassigned column j; the dual variable $u[i]$ corresponds to row i and $v[j]$ to column j. We denote the reduced costs by: $\mathrm{cred}[i,j]=c[i,j]-u[i]-v[j]$, and sometimes we refer to the dual variables as "prices".

2. A Review of Assignment Algorithms

Methods to solve the LAP can be classified in three categories, that are based on algorithms for

a) maximum flow,
b) shortest paths,
c) linear programming.

Most *algorithms based on maximum flow* are primal-dual methods. For an introduction to these methods we refer to Papadimitriou and Steiglitz [26]. The Hungarian algorithm of Kuhn [22] actually served as the algorithm from which the general primal-dual algorithm was derived. The original method has computational complexity $O(n^4)$, but later $O(n^3)$ versions were developed (Lawler [23]). Jonker and Volgenant [20] give some simple, but effective improvements.

Bertsekas [3] also presents a primal-dual algorithm. The method is Hungarian-type, and the best version even switches to the Hungarian algorithm itself as soon as the original method becomes less effective. We describe part of it in Section 4.

The *methods based on shortest paths* are dual algorithms: dual feasibility exists and primal feasibility has to be reached. This is achieved by considering the LAP as a minimum cost flow problem, solved by steps that involve finding shortest paths on an auxiliary graph.

In this group two algorithms, both of complexity $O(n^3)$, stand out: Hung and Rom's [18] and Tomizawa's [29]. The former is the more ingenious, but the latter the fastest, as shown in Section 8. Tomizawa augments partial assignments into a complete solution by primal steps in each of which one shortest augmenting path is determined. Hung and Rom's initial solution is complete, but may be infeasible. They determine in each step a shortest path tree, which takes more effort, but may contain more disjoint augmenting paths. We consider shortest augmenting path LAP algorithms separately in Section 3.

The so-called Bradford method of Mack [24] is also of interest, especially for its intuitively appealing presentation. As originally presented, it has computational complexity $O(n^4)$. The method is equivalent to the Hungarian algorithm; adapting it to obtain complexity $O(n^3)$ results in an algorithm close to Tomizawa's (Jonker [19]).

Good results on sparse LAPs are obtained by Carraresi and Sodini [7] with an algorithm based on the shortest path method of Glover et al. [15, 16].

The *linear programming based algorithms* in the third category are specialized versions of the simplex method. The best published results are from Barr, Glover and Klingman [2]. A major difficulty with these methods is the phenomenon of zero pivot steps. A drawback is their relatively complex implementation as compared to the other approaches. Computational experiments (Hung and Rom [18], McGinnis [25]) show that these algorithms are outperformed by the best primal-dual and dual methods.

The $O(n^3)$ signature algorithm presented by Balinski [1] and analyzed by Goldfarb [17] also belongs to this category. It considers feasible dual solutions corresponding to trees in the bipartite graph of row and column nodes. The algorithm can be considered a variant of the Hungarian method. Nothing definite is known yet about its computational performance.

3. Shortest Augmenting Paths Based Algorithms

Linear assignment is a special case of minimum cost flow, for which an algorithm exists called "Buildup" in Papadimitriou and Steiglitz [26]. Ford and Fulkerson [14] attribute this method to Jewell (1959) and to Busacker and Gowen (1961). It uses flow augmentation along paths in an auxiliary network, that, depending on the current flow, can be constructed from the original one.

Tomizawa [29] noted that shifting from the original costs of the assignment problem to the (non-negative) reduced costs allows the algorithm of Dijkstra [12] to solve the shortest path problems. With $O(n)$ flow augmentations, this leads to an $O(n^3)$ computational complexity of the algorithm. The theoretical improvements for minimum cost flow algorithms of Edmonds and Karp [13] can also be applied to the assignment problem. The resulting method is equivalent to that of Tomizawa.

In the original version of Tomizawa an augmentation step consists of finding a shortest augmenting path with both initial row and final column specified.